Counting statistics for the Anderson impurity model: Bethe ansatz and Fermi liquid study

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Abstract

We study the counting statistics of charge transport in the Anderson impurity model (AIM) employing both Keldysh perturbation theory in a Fermi liquid picture and the Bethe ansatz. In the Fermi liquid approach, the object of our principal interest is the generating function for the cumulants of the charge current distribution. We derive an *exact analytic formula* relating the full counting statistic (FCS) generating function to the self-energy of the system in the presence of a measuring field. We first check that our approach reproduces correctly known results in simple limits, like the FCS of the resonant level system (AIM without Coulomb interaction). We then proceed to study the FCS for the AIM perturbatively in the Coulomb interaction. By comparing this perturbative analysis with a strong coupling expansion, we arrive at a conjecture for an expression for the FCS generating function at $\mathcal{O}(V^3)$ (V is the voltage across the impurity) valid at all orders in the interaction.

In the second part of the article, we examine a Bethe ansatz analysis of the current noise for the AIM. Unlike the Fermi liquid approach, here the goal is to obtain qualitative, not quantitative, results for a wider range of voltages both in and out of a magnetic field. Particularly notable are finite field results showing a double peaked structure in the current noise for voltages satisfying $eV \sim \mu_B H$. This double peaked structure is the "smoking gun" of Kondo physics in the current noise and is directly analogous to the single peak structure predicted for the differential conductance of the AIM.

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I. INTRODUCTION

The subject of counting statistics is rooted in the historical paper by Schottky [1] where the measurements of charge noise have been carried out and interpreted as the basis for determining the elementary charge e of the current carriers: electrons. Contemporary transport experiments are being performed on nano-structures, usually involving two electron reservoirs (left and right) and a central constriction [2]. The mean electric current, or linear conductance is well understood in terms of scattering theory [3] and, for a single conducting channel, is given by the Landauer formula:

$$G_0 = \frac{2e^2}{h} T_0 \ , \tag{1}$$

where T_0 is the transmission coefficient and factor 2 stems from the electron spin. However, due to the quantum nature of the problem, the current is bound to fluctuate. In particular, this gives rise to interesting noise (the second moment of the current distribution) properties extensively discussed in the literature [4].

With the third moment of the current distribution now available experimentally [5], it is natural to widen the question to the full current distribution function or the full counting statistics (FCS). One way to formulate this question is to ask what is the probability $P(Q)$ that charge Q will be transmitted through the system during the waiting time $\mathcal{T}$ and for a given bias voltage V. As electrons are discrete particles, a naive guess at $P(Q)$ would be the Poisson's distribution:

$$P(Q) = \frac{\langle Q \rangle^Q}{Q!} e^{-\langle Q \rangle} \ , \tag{2}$$

where $\langle Q \rangle = G_0 V \mathcal{T}$. For simplicity, we set $e = \hbar = 1$ in what follows. The electrons, however, are not only discrete particles but also quantum particles obeying Fermi-Dirac statistics. Due to the Pauli principle the electrons will tunnel 'one by one'. So, given the 'number of attempts', $N = V\mathcal{T}/\pi$, one would expect the total probability be proportional to the probability of successes T_0^Q as well as the probability of failures $(1 - T_0)^{N-Q}$. The resulting probability distribution is binomial:

$$P(Q) = \left[\begin{array}{c} Q \\ N \end{array} \right] T_0^Q (1 - T_0)^{N-Q} \ , \tag{3}$$

where the binomial coefficient in front simply follows from the normalization: $\sum_Q P(Q) = 1$. Note that the binomial distribution is a clear signature of Fermi statistics; indeed, the re-

spective probability distribution for bosons is the inverse binomial [6]. In practice, it is more convenient to work with the generating function $\chi(\lambda) = \sum_Q e^{i\lambda Q} P(Q)$, where the Fourier transform variable λ is called the 'counting field' (see below). The irreducible moments of the charge distribution immediately follow [7]:

$$\ln \chi(\lambda) = \sum_{n=1}^{\infty} \langle\langle Q^n \rangle\rangle \frac{(i\lambda)^n}{n!} \; . \tag{4}$$

The generating function for the binomial distribution is simply:

$$\chi_{\rm binomial} = [1 + T_0(e^{i\lambda} - 1)]^N \; . \tag{5}$$

From this equation one easily recovers the Landauer formula, $\langle Q \rangle = NT_0$, the well known expression for the shot noise $\langle\langle Q^2 \rangle\rangle = NT_0(1-T_0)$, and obtains the following expression for the third moment: $\langle\langle Q^3 \rangle\rangle = NT_0(1-T_0)(1-2T_0)$. Note that for low transmission ($T_0 \to 0$), the statistics reverts to Poissonian, while for perfect transmission ($T_0 \to 1$), there are no current fluctuations and $\chi(\lambda) = i\lambda N$. The physics described so far has been understood in the seminal paper by Levitov and Lesovik [8] (see also [9]), where they derive a more general formula for the generating function

$$\begin{aligned} \ln \chi_0(\lambda; V; \{T(\omega)\}) = 2\mathcal{T} \int \frac{d\omega}{2\pi} \ln \Big\{ 1 + T(\omega) \\ \times \; \big[n_L(1-n_R)(e^{i\lambda} - 1) + n_R(1-n_L)(e^{-i\lambda} - 1) \big] \Big\} , \end{aligned} \tag{6}$$

which is valid for finite voltage, temperatures, and allows for the energy dependent transmission coefficient. Here $n_{L/R}(\omega) = n_F(\omega \mp V/2)$ are the thermal electron distributions in the left and right leads and $n_F(\omega)$ is the Fermi function. Clearly T_0 is $T(\omega)$ at the Fermi energy, set at $\omega = 0$. Schönhammer has recently re-examined this formula, Eq.(6), by an alternative method, and found that it is correct [10].

The discussion so far has focused upon *non-interacting* electrons. But while counting statistics for non-interacting electrons is by now comprehensively understood, the same cannot be said when the electrons interact with each other and with the substrate. Consequently the understanding of the interaction effects on the FCS has become an important issue. There has been many papers on the subject in recent years with many interesting yet miscellaneous results. Certainly no general paradigm as to how interactions should affect the statistics has as yet emerged. Misunderstandings dominating the subject only a

short time ago are well illustrated by the following example. The generating function $\chi(\lambda)$ for Matveev's Coulomb blockade setup [11] (equivalent to the $g = 1/2$ Kane and Fisher problem [12], which is in turn equivalent to the $\alpha = 1/2$ dissipation problem first solved by Guinea [13]) has been calculated independently by three different methods in [14], [15], and [16] with seemingly very different results. It was only understood later that all three results are indeed correct and represent one and the same function (see Appendix C to Ref. [17]). Moreover the distribution in question turned out both to be simple and to represent a particular case of Eqn. (6) with a specific choice of the transmission coefficient, $T(\omega)$. This lack of a coherent picture of the FCS in strongly correlated systems is, we believe, simply explained. As is illustrated by the prominence of the Fermi and Luttinger liquid paradigms, it is accepted that in the condensed matter it is the *low-energy* physics which is universal. The FCS is no exception. On an energy scale set by the bare parameters, it is therefore the low-temperature, low-voltage expansions of $\chi(\lambda)$ where universal results are to be found. The high-voltage (temperature) distributions may be enormously fascinating but are destined to remain model dependent.

In this paper we collect together a number of such universal results, presented in two parts. In the first part, Sections II and III, we study the generating function, χ, using Keldysh perturbation theory in a Fermi liquid approach. In Section II, we introduce the Keldysh method for calculating the statistics. In the process we establish an exact relationship between the generating function and the self energy. In Section III, we study the FCS for the AIM both in perturbation theory and in the strong coupling limit. By comparing the two we propose a conjecture for χ at low voltages, i.e. $\mathcal{O}(V^3)$, but valid at all orders of the interaction.

In the second part of the paper, Sections IV through VI, we switch tacts and instead employ a Bethe ansatz analysis of the current moments for the AIM. We however limit ourselves to exploring the behavior of the current and the current noise in the AIM's Kondo regime. Our results for these quantities differ from those of the first section of this paper. Here the focus is on their qualitative not quantitative features but over a larger range of voltages (though still much smaller than any bare energy scale) and for finite magnetic fields. In Section IV we review the Bethe Ansatz method for calculating the current and noise. We then present results for zero magnetic field in Section V where, in addition, a comparison is made to Fermi liquid calculations. In the final section, Section VI, we consider

the properties of the noise in finite magnetic fields. Here is found the most significant result of the second part of the paper. We are argue that in the vicinity of voltages commensurate with the magnetic field, the current noise should see a double humped enhancement. This enhancement is the analog of that seen in the current when $eV \sim \mu_B H$ and so represents a 'smoking gun' [46] of Kondo physics.

II. KELDYSH METHOD FOR THE CALCULATION OF CURRENT STATISTICS: GENERAL RESULTS

The calculation of the charge statistics is usually accomplished by coupling the system to a 'measuring device'. In the original gedanken experiment by Levitov and Lesovik it is a fictitious spin-1/2 galvanometer coupled to the current [9]. The transmitted charge is then proportional to the change of the spin phase. As has been shown by Nazarov [18], the counting of charge can in general be done by coupling the system to a fictitious field and calculating the non-linear response, so leading to the same results. In fact the standard quantum mechanical formula, $P(Q) = \langle \delta(\hat{Q} - Q) \rangle$, can also be used provided that the central region is initially decoupled from the leads.

According to [19] the generating function is given by the following average,

$$\chi(\lambda) = \left\langle T_{\rm C} \exp\left[-i \int\limits_{\rm C} T_\lambda(t) dt \right] \right\rangle \ , \tag{7}$$

where C is the Keldysh contour [20], $\lambda(t)$ is the *measuring field* which is non-zero only during the measuring time $\mathcal{T}$: $\lambda(t) = \lambda\theta(t)\theta(\mathcal{T} - t)$ on the forward path and $\lambda(t) = -\lambda\theta(t)\theta(\mathcal{T} - t)$ on the backward path. Introducing the operator T_R transferring electrons through the system in the direction of the current, as well as its counterpart T_L, we can write

$$T_\lambda = e^{i\lambda(t)/2} T_R + e^{-i\lambda(t)/2} T_L \ . \tag{8}$$

Note that $T_R^\dagger = T_L$ in any system. Consequently, writing out (7) explicitly in terms of the time–ordered and anti–time–ordered products, one arrives at the conjugation property,

$$\chi^*(\lambda) = \chi(-\lambda) \ . \tag{9}$$

We now allow $\lambda(t)$ to be an arbitrary function on the Keldysh contour, $\lambda_\pm(t)$ on the forward/backward path. Then a generalised expression for the generating function is

$$\chi[\lambda_-(t), \lambda_+(t)] = \langle T_{\rm C} e^{-i \int_{\rm C} dt\, T_\lambda(t)} \rangle \ . \tag{10}$$

Assume that the measuring field changes only very slowly in time. Then one can write

$$\chi[\lambda_-(t),\lambda_+(t)] = \exp\left[-i\int_0^{\mathcal{T}} \mathcal{U}[\lambda_-(t),\lambda_+(t)]dt\right], \tag{11}$$

where $\mathcal{U}(\lambda_-,\lambda_+)$ is the *adiabatic potential*. Note that Eq.(11) captures the leading, linear in $\mathcal{T}$, contribution to the phase. For a truly slow-varying measuring field (analytic in t), the corrections are exponentially small in $\mathcal{T}$. Even if the function $\lambda(t)$ has isolated jumps, Eq.(11) still holds at large $\mathcal{T}$, but the corrections are then of the order of $\ln \mathcal{T}$. This is in full analogy to the non-equilibrium version [21] of the old X-ray problem. Once the adiabatic potential is known, the statistics is given by

$$\ln \chi(\lambda) = -i\mathcal{T}\,\mathcal{U}(\lambda,-\lambda)\ . \tag{12}$$

Alternatively one can level off the $\lambda_\pm$ functions in Eq.(10) to different constants as

$$\chi[\lambda_-(t),\lambda_+(t)] \to \chi(\lambda_-,\lambda_+)\ , \tag{13}$$

then $\chi(\lambda) = \chi(\lambda,-\lambda)$. Note that the conjugation property (9) now generalises to

$$\chi^*(\lambda_-,\lambda_+) = \chi(\lambda_+,\lambda_-)\ , \tag{14}$$

or

$$\mathcal{U}^*(\lambda_-,\lambda_+) = -\mathcal{U}(\lambda_+,\lambda_-)\,. \tag{15}$$

To compute the adiabatic potential we observe that according to the non-equilibrium version of the the Feynman–Hellmann theorem [21],

$$\frac{\partial}{\partial\lambda_-}\mathcal{U}(\lambda_-,\lambda_+) = \left\langle \frac{\partial T_\lambda(t)}{\partial\lambda_-}\right\rangle_\lambda\ , \tag{16}$$

where the averages are defined as

$$\langle A(t)\rangle_\lambda = \frac{1}{\chi(\lambda_-,\lambda_+)}\left\langle T_{\rm C}\left\{A(t)e^{-i\int_{\rm C} T_\lambda(t)dt}\right\}\right\rangle \tag{17}$$

(and similarly for multi–point functions) where λ's are understood to be *different* on the two time branches. Note that the above one–point averages depend on the branch the time t is on (though not on the value of t on that branch):

$$\langle A(t_-)\rangle_\lambda \neq \langle A(t_+)\rangle_\lambda\,. \tag{18}$$

Therefore the average in Eq. (16) *must* be taken on the forward branch of the Keldysh contour. An advantage of this Hamiltonian approach is that the calculation of the adiabatic potential $\mathcal{U}$ is reduced to a calculation of Green's functions (GF), albeit non-equilibrium ones. So we can use the well developed diagram technique (and relate to many known results within this method) without being restricted to the scattering problem.

The conventional test model to consider is the Anderson impurity model (see [22] for a review). The Hamiltonian of the model consists of three contributions,

$$H = H_0 + H_T + H_C \,. \tag{19}$$

The kinetic part

$$H_0 = \sum_\sigma H_0[\psi_{R/L,\sigma}] + \sum_\sigma (\Delta_0 + \sigma h) d_\sigma^\dagger d_\sigma \,, \tag{20}$$

describes a single fermionic level (which we shall also call the 'dot') with electron creation operators, $d_\sigma^\dagger$ (σ is the spin index), energy, Δ_0, and subject to a local magnetic field, h. The electrons in the two non-interacting metallic leads, $i = R, L$, are represented by $\psi_{R/L,\sigma}$. The leads and the dot are coupled via the tunnelling operator,

$$H_T = \sum_\sigma \left[\gamma_L e^{i\lambda(t)/2} d_\sigma^\dagger \psi_{L\sigma} + \gamma_R \psi_{R\sigma}^\dagger d_\sigma \psi_{R\sigma}^\dagger + \text{H.c.} \right] \,, \tag{21}$$

with, in general, different amplitudes, $\gamma_{R,L}$. We have included the counting field into the Hamiltonian (left junction). One can as well incorporate it into the right junction (or both), the results are the same due to the gauge symmetry of the Hamiltonian. Finally, we include the Coulomb repulsion on the dot,

$$H_C = U n_\uparrow n_\downarrow \,, \tag{22}$$

where $n_\sigma = d_\sigma^\dagger d_\sigma$. The bias voltage V is incorporated into the full Hamiltonian by assuming different chemical potentials in the leads: $\mu_L - \mu_R = V \geq 0$.

It is useful to define two auxiliary GFs,

$$\begin{aligned} F_\lambda(t,t') &= -i\langle T_{\rm C}\{\psi_L(t) d^\dagger(t')\}\rangle_\lambda \\ \widetilde{F}_\lambda(t,t') &= -i\langle T_{\rm C}\{d(t)\psi_L^\dagger(t')\}\rangle_\lambda \,. \end{aligned} \tag{23}$$

Hence the derivative of the adiabatic potential is given by

$$\begin{aligned} \frac{\partial}{\partial\lambda_-}\mathcal{U}(\lambda_-,\lambda_+) &= \frac{\gamma_L}{2} \lim_{\epsilon\to 0+} \int \frac{d\omega}{2\pi} e^{i\epsilon\omega} \\ &\times \left[e^{i\lambda_-/2} F_\lambda^{--}(\omega) - e^{-i\lambda_-/2} \widetilde{F}_\lambda^{--}(\omega) \right] \,. \end{aligned} \tag{24}$$

As is standard, the mixed GF's can be written as combinations of bare lead GF's and an exact impurity GF, $D(t,t')$,

$$\begin{aligned}\widetilde{F}_\lambda(t,t') &= \int_{\rm C} dt'' g_L(t-t'')e^{-i\lambda(t'')}D(t'',t')\,,\\ F_\lambda(t,t') &= \int_{\rm C} dt'' D(t,t')e^{-i\lambda(t'')}g_L(t''-t')\,.\end{aligned} \tag{25}$$

Plugging this back into Eq.(24) one obtains

$$\begin{aligned}\frac{\partial}{\partial\lambda_-}\mathcal{U}(\lambda_-,\lambda_+) = \frac{\gamma_L^2}{2}\int\frac{d\omega}{2\pi}\Big[&e^{-i\bar\lambda/2}D^{-+}g_L^{+-}\\ &-e^{i\bar\lambda/2}g_L^{-+}D^{+-}\Big]\,,\end{aligned} \tag{26}$$

where $\bar\lambda = \lambda_- - \lambda_+$. Thus the problem is now reduced to calculation of the impurity GF.

To illustrate how the method works we start with the trivial case of the resonant level model: $U = 0$. Using the GFs of the lead electrons (see, for example, Ref. [16]),

$$\begin{aligned}g_i^{--}(\omega) &= g_i^{++}(\omega) = i2\pi\rho_0[n_i - 1/2]\,,\\ g_i^{-+}(\omega) &= i2\pi\rho_0 n_i\,,\\ g_i^{+-}(\omega) &= -i2\pi\rho_0[1-n_i]\,,\end{aligned} \tag{27}$$

where ρ_0 is the density of states in the electrodes in the vicinity of Fermi level, one easily obtains the bare impurity GF (we use the original notation of Keldysh for the GFs because the standard identity $g^{--}+g^{++} = g^{-+}+g^{+-}$ is spoiled by the measuring field and the Keldysh rotation is useless):

$$\hat D_0^{-1}(\omega) = \begin{bmatrix}\omega-\Delta_0-i\Gamma_L(2n_L-1)-i\Gamma_R(2n_R-1) & 2ie^{i\bar\lambda/2}\Gamma_L n_L + 2i\Gamma_R n_R\\ -2ie^{-i\bar\lambda/2}\Gamma_L(1-n_L)-2i\Gamma_R(1-n_R) & -\omega+\Delta_0-i\Gamma_L(2n_L-1)-i\Gamma_R(2n_R-1)\end{bmatrix},$$

where $\Gamma_{R,L} = (\pi\rho_0\gamma_{R,L})^2$. Inverting this matrix results in

$$\begin{aligned}\hat D_0(\omega) &= \frac{1}{\mathcal{D}_0(\omega)}\\ &\times\begin{bmatrix}\omega-\Delta_0+i\Gamma_L(2n_L-1)+i\Gamma_R(2n_R-1) & 2ie^{i\bar\lambda/2}\Gamma_L n_L + 2i\Gamma_R n_R\\ -2ie^{-i\bar\lambda/2}\Gamma_L(1-n_L)-2i\Gamma_R(1-n_R) & -\omega+\Delta_0+i\Gamma_L(2n_L-1)+i\Gamma_R(2n_R-1)\end{bmatrix},\end{aligned}$$

where $\Gamma = \Gamma_R + \Gamma_L$ and we call the object

$$\mathcal{D}_0(\omega) = (\omega-\Delta_0)^2 + \Gamma^2 + 4\Gamma_L\Gamma_R\Big[n_L(1-n_R)(e^{i\bar\lambda/2}-1) + n_R(1-n_L)(e^{-i\bar\lambda/2}-1)\Big]\,, \tag{28}$$

the 'Keldysh denominator'. On the technical side, while this object is a smooth function of energy in the standard technique (expressible via the advanced and retarded GF's), here it has discontinuities at the Fermi levels. Inserting this result into Eq.(26) and integrating over λ_- (which can be done in a closed form) leads to the formula (6) with the Breit-Wigner transmission coefficient,

$$T(\omega) = \frac{4\Gamma_L\Gamma_R}{(\omega-\Delta_0)^2+\Gamma^2}\,, \tag{29}$$

as expected for the resonant level problem.

Turning to the interacting case, we define the impurity self-energy in the standard fashion [20]:

$$\hat{D}(\omega) = \hat{D}_0(\omega) + \hat{D}(\omega)\hat{\Sigma}(\omega)\hat{D}_0(\omega)\,. \tag{30}$$

Substituting this into (26) one obtains a *general formula* for the statistics in interacting one-channel systems

$$\begin{aligned}\frac{\partial}{\partial\lambda_-}\mathcal{U}(\lambda_-,\lambda_+) \;=\; -\Gamma_L\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi\mathcal{D}(\omega)}\Big\{2\Gamma_R\left[e^{i\bar{\lambda}/2}n_L(1-n_R)-e^{-i\bar{\lambda}/2}n_R(1-n_L)\right] \\ -\;i\left[e^{i\bar{\lambda}/2}n_L\Sigma^{+-}+e^{-i\bar{\lambda}/2}(1-n_L)\Sigma^{-+}\right]\Big\}\,,\end{aligned} \tag{31}$$

that expresses the generating function in terms of the (λ-dependent) impurity self-energy. Here $\mathcal{D}(\omega)$ is the determinant of the (inverse) interacting impurity GF. For $\bar{\lambda}=0$ this equation yields the electric current and can be shown to reproduce the result by Meir and Wingreen [23]. Formula (31) is not restricted to the Anderson model but is applicable for any similar one-channel systems (for example, one can add on the electron–phonon interaction or consider a double dot).

The general formula (31) allows us to investigate the important limit: linear response statistics at zero temperature. Indeed let us take a closer look at the general formula. An important technical observation is that the limits $V\to 0$ and $\omega\to 0$ do not commute in the presence of the counting field. Indeed, calculating the Keldysh determinant in both limits we see that even for the non-interacting case

$$\lim_{\omega\to 0}\lim_{V\to 0}\mathcal{D}_0(\omega,V,\lambda) = \Delta_0^2+\Gamma^2\,, \tag{32}$$

but

$$\lim_{V\to 0}\lim_{\omega\to 0}\mathcal{D}_0(\omega,V,\lambda) = \Delta_0^2+\Gamma^2+4\Gamma_L\Gamma_R(e^{i\lambda}-1)\,. \tag{33}$$

The latter scheme needs to be implemented when analyzing the first term in Eq.(31) in the linear response limit, as the energy integration here is restricted to $[0, V]$. This leads to a transmission coefficient type contribution to the generating function (see below). In the second term in Eq.(31), however, the integration over ω is not restricted to $[0, V]$. But due to Auger type effects [24] one expects that there are contributions to the current (and FCS) at all energies. This effect is itself proportional to the applied voltage and only leads to non-linear corrections to the FCS. Hence the energy integration can in fact be regarded as restricted to $[0, V]$ even in the second term in Eq.(31). Since the self-energy does not have external lines and all the internal frequencies have to be integrated over, the limits $V \to 0$ and $\omega \to 0$ in this case commute. That means that for the evaluation of the self-energy to the lowest order in V one is allowed to use the equilibrium GFs, calculated in presence of the counting field λ, i. e. (28) with $n_R = n_L = n_F$ and with the corresponding Keldysh denominator. Therefore all diagonal Keldysh GFs are equal to those in the equilibrium and all off-diagonal ones are simply proportional to the same diagrams as in equilibrium. Since any given off-diagonal self-energy diagram describes an inelastic process, it should vanish for $\omega \to 0$ and we arrive at a conclusion that

$$\lim_{\omega \to 0} \hat{\Sigma}(\omega) = \mathrm{Re}\, \Sigma^R(0) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{34}$$

even at finite λ. Eq.(31) thus leads to the important result

$$\ln \chi(\lambda) = N \ln \left\{ 1 + \frac{\Gamma^2}{[\mathrm{Re}\Sigma^{(R)}(0)]^2 + \Gamma^2} (e^{i\lambda} - 1) \right\} , \tag{35}$$

or to $\ln \chi(\lambda) = i\lambda N$ for the symmetric Anderson impurity model. In the case of an *asymmetrically coupled* impurity, $\Gamma_R \neq \Gamma_L$, the numerator of (35) changes to $\Gamma_R \Gamma_L$ while the denominator contains $(\Gamma_R + \Gamma_L)/2$ instead of Γ.

The result (35) allows simple generalisations to asymmetric systems in a magnetic field h. According to [25] the real part of the self-energy is given by

$$\mathrm{Re}\Sigma^{(R)}_\sigma(0) = \chi_{\mathrm{c}} \kappa + \sigma \chi_{\mathrm{s}} h \ ,$$

where $\chi_{\mathrm{c/s}}$ are exact charge/spin susceptibilities and $\kappa \sim \Delta_0 + U/2$ is a particle–hole symmetry breaking field. Consequently

$$\ln \chi(\lambda) = \frac{N}{2} \ln \left\{ \left[1 + \frac{\Gamma^2}{[\chi_{\mathrm{c}} \kappa + \chi_{\mathrm{s}} h]^2 + \Gamma^2} (e^{i\lambda} - 1) \right] \times \left[1 + \frac{\Gamma^2}{[\chi_{\mathrm{c}} \kappa - \chi_{\mathrm{s}} h]^2 + \Gamma^2} (e^{i\lambda} - 1) \right] \right\} . \tag{36}$$

An advantage of this formula is that the susceptibilities are known *exactly* from the Bethe-Ansatz results [26, 27]. We stress that the result (35) is not limited to the Anderson model but will hold for any similar model, hence the *binomial theorem*. It is clear in hindsight that all the non-elastic processes fall out in the $T = 0$ linear response limit. Still it is a remarkable result that *all* the moments have a simple expression in terms a single number: the effective transmission coefficient. The binomial distribution is universal as long as systems with a single conducting channel are concerned.

III. FCS FOR THE ANDERSON MODEL

A. Perturbative expansion in the Coulomb interaction

We now proceed with the perturbative expansion in the Coulomb interaction U. The self-energy, up to U^2-order, is given, in the time domain, by

$$\hat{\Sigma}(t) = \begin{bmatrix} -iUD_0^{--}(0) + U^2[D_0^{--}(t)]^2 D_0^{--}(-t) & -U^2[D_0^{-+}(t)]^2 D_0^{+-}(-t) \\ -U^2[D_0^{+-}(t)]^2 D_0^{-+}(-t) & iUD_0^{++}(0) + U^2[D_0^{++}(t)]^2 D_0^{++}(-t) \end{bmatrix} . \quad (37)$$

We restrict the calculation to the case of the particle-hole symmetric Anderson model $\Delta_0 = -U/2$,. It can be shown that the contribution to the statistics linear in U vanishes in the symmetric case. We therefore concentrate now on the second-order correction.

The equilibrium self-energy is, in fact, known to all orders in U [25]:

$$\hat{\Sigma}_{\rm eq}(\omega) = (1-\chi_{\rm e})\omega \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{i\chi_{\rm o}^2}{2\Gamma}\omega^2 \begin{bmatrix} \mathrm{sign}(\omega) & 2\theta(-\omega) \\ -2\theta(\omega) & \mathrm{sign}(\omega) \end{bmatrix} , \quad (38)$$

where $\chi_{\rm e/o}$ are the *exact* even–odd susceptibilities (i.e. correlations of $n_\uparrow$ with $n_\uparrow$ and $n_\downarrow$ respectively), which in weak coupling expand as:

$$\chi_{\rm e} = 1 + \left(3 - \frac{\pi^2}{4}\right)\frac{U^2}{\pi^2\Gamma^2} + \dots , \quad \chi_{\rm o} = -\frac{U}{\pi\Gamma} . \quad (39)$$

For finite V and λ in the region $-V/2 < \omega < V/2$ we find:

$$\begin{aligned} \hat{\Sigma}(\omega) \;=\; & (1-\chi_{\rm e})\omega \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (40) \\ & - \frac{iU^2}{8\pi^2\Gamma^3} \begin{bmatrix} 6\omega V & e^{-i\lambda}\left(\frac{3V}{2}-\omega\right)^2 + 3\left(\frac{V}{2}-\omega\right)^2 \\ -e^{-2i\lambda}\left(\frac{3V}{2}+\omega\right)^2 - 3e^{-i\lambda}\left(\frac{V}{2}+\omega\right)^2 & 6\omega V \end{bmatrix} . \end{aligned}$$

Further, for $\omega > V/2$ one obtains

$$\Sigma^{-+}(\omega) = -\frac{ie^{-i\lambda}U^2}{8\pi^2\Gamma^3}\left(\frac{3V}{2}-\omega\right)^2\theta\left(\frac{3V}{2}-\omega\right), \tag{41}$$

$$\tag{42}$$

while for $\omega < V/2$ the following holds:

$$\Sigma^{+-}(\omega) = \frac{ie^{-2i\lambda}U^2}{8\pi^2\Gamma^3}\left(\frac{3V}{2}+\omega\right)^2\theta\left(\frac{3V}{2}+\omega\right). \tag{43}$$

Substituting these self-energies into (31), expanding around the perfect transmission (hence the sign change of λ), and formally expressing the result in terms of the susceptibilities, one finds the following formula:

$$\ln\chi(\lambda) = N\left\{i\lambda + \frac{V^2}{3\Gamma^2}\left[\frac{\chi_{\rm e}^2+\chi_{\rm o}^2}{4}(e^{-i\lambda}-1)+\frac{\chi_{\rm o}^2}{2}(e^{-2i\lambda}-1)\right]\right\}+O(V^4)\,. \tag{44}$$

This formula is *only valid at the order* U^2. There are, however, reasons to think that it might be exact (see below). It is tempting to interpret the appearance of the double exponential terms as an indication of a coherent tunnelling of electron pairs (caution: similar terms would also appear for the non-interacting resonant-level model due to the energy dependence of the transmission coefficient).

B. Strong coupling expansion

In the opposite limit of large U, the Schrieffer–Wolf type transformation [28], tailored to the lead geometry [29], can be applied and results in a Kondo type model. For the latter model in the strong–coupling limit, when the spin on the dot is absorbed into the Fermi sea forming a singlet, Nozières [30] devised a Landau–Fermi–liquid description based on a 'molecular field' expansion of the phase shift of the s–wave electrons:

$$\delta_\sigma(\varepsilon) = \delta_0 + \alpha\varepsilon + \phi^a(n_\sigma - n_{\bar\sigma})\,, \tag{45}$$

where $\delta_0 = \pi/2$, α, and ϕ^a are phenomenological parameters corresponding to the residual potential scattering and the residual interactions, respectively. These processes are generated by polarizing the Kondo singlet and so are of the order $\sim 1/T_K$, where T_K is the Kondo temperature. The specific heat coefficient is proportional to $\alpha/(\pi\nu)$ while the magnetic susceptibility is proportional to the sum $\alpha/(\pi\nu) + 2\phi^a/\pi$. Simple arguments were advanced

in Ref. [30] to the effect that, because the Kondo singularity is tied up to the Fermi level, there exists a relation $\alpha = 2\nu\phi^a$ between the two processes in Eq. (45). In particular, this explains why the Wilson ratio is equal to 2. The strong–coupling Hamiltonian that describes the scattering and interaction processes encoded in Nozières Eq. (45) is of the form $H = H_0 + H_{\rm sc} + H_{\rm int}$. The free Hamiltonian here is

$$H_0 = \sum_{p,\sigma} \varepsilon_p (c^\dagger_{p,\sigma} c_{p,\sigma} + a^\dagger_{p,\sigma} a_{p,\sigma}) + VQ \ , \tag{46}$$

where $c^\dagger$ is the creation operator for the s–wave electrons, $a^\dagger$ is the creation operator of the p–wave electrons, included in order to account for the transport [31], and the operator

$$Q = \frac{1}{2} \sum_{p,\sigma} (c^\dagger_{p,\sigma} a_{p,\sigma} + a^\dagger_{p,\sigma} c_{p,\sigma})$$

stands for the (minus) charge transferred across the junction. The scattering term is

$$H_{\rm sc} = \frac{\alpha}{2\pi\nu T_K} \sum_{p,p',\sigma} (\varepsilon_p + \varepsilon_{p'}) c^\dagger_{p,\sigma} c_{p',\sigma} \ , \tag{47}$$

while the interaction term reads

$$H_{\rm int} = \frac{\phi}{\pi\nu^2 T_K} c^\dagger_\uparrow c_\uparrow c^\dagger_\downarrow c_\downarrow \ , \tag{48}$$

where $c_\sigma = \sum_p c_{p,\sigma}$ and we have changed to the dimensionless amplitudes α and ϕ, so that in the actual Kondo model $\alpha = \phi = 1$ (in the intermediate calculations it is convenient to treat α and ϕ as free parameters though). By the nature of the strong–coupling fixed point, the operators α and ϕ are irrelevant in the renormalization group sense and therefore the perturbative expansion in α and ϕ is expected to converge.

The shot noise in this model was recently discussed in Refs. [32, 33]. We now turn to the FCS. To this end we introduce the the measuring field, which couples, in the Lagrangian formulation, to the current via a term in the action $\int dt \lambda(t) \dot{Q}(t) = -\int dt \dot{\lambda}(t) Q(t)$ that can be gauged away by the canonical transformation

$$\begin{aligned} c \to c_\lambda &= \cos(\lambda/4) c - i \sin(\lambda/4) a \ , \\ a \to a_\lambda &= -i \sin(\lambda/4) c + \cos(\lambda/4) a \ . \end{aligned} \tag{49}$$

We therefore reach the conclusion that the charge measuring field enters this problem as a rotation of the strong–coupling basis of the s– and the p–states. While H_0 is invariant

under this substitution, it should be performed in both the scattering and the interaction Hamiltonians, $H_{\rm sc}[c]+H_{\rm int}[c]\to H_\lambda=H_{\rm sc}[c_\lambda]+H_{\rm int}[c_\lambda]$, when calculating the statistics. It is easily checked that at the first order in λ: $H_\lambda=H_{\rm sc}+H_{\rm int}+(\lambda/4)\hat{I}_{\rm bs}+O(\lambda^2)$, where $\hat{I}_{\rm bs}$ is the backscattering current operator

$$\begin{aligned}\hat{I}_{\rm bs} = &- i\frac{\alpha}{4\pi\nu T_K}\sum_{p,p',\sigma}(\varepsilon_p+\varepsilon_{p'})(c^\dagger_{p,\sigma}a_{p',\sigma}-a^\dagger_{p,\sigma}c_{p',\sigma})\\ &- i\frac{\phi}{2\pi\nu^2 T_K}\sum_\sigma(c^\dagger_\sigma a_\sigma - a^\dagger_\sigma c_\sigma)c^\dagger_{\bar\sigma}c_{\bar\sigma}\ ,\end{aligned}\tag{50}$$

alternatively available from the commutator $\hat{I}_{\rm bs}=-\dot{Q}=i[Q,H]$.

Applying the standard linked cluster expansion (still valid on the Keldysh contour, of course) [34], we see that the leading correction to the distribution function is given by a connected average

$$\ln\chi(\lambda)=iN\lambda-\frac{1}{2}\int_C dt_1dt_2\langle T_C\{H_\lambda(t_1)H_\lambda(t_2)\}\rangle_{\rm c}+...\tag{51}$$

The neglected terms α^4, $\alpha^2\phi^2$, ϕ^4, etc., are of the higher order in voltage (temperature) than the main correction because of the irrelevant nature of the perturbation. In order to make progress with Eq. (51), one only needs the Green's function of the λ–rotated c–operator, which is easily seen to be the following matrix in Keldysh space:

$$\begin{aligned}\hat{g}_\lambda(p,\omega) \ = \ & i\pi\delta(\varepsilon_p-\omega)\,\{\,[f(\omega-V/2)+f(\omega-V/2)-1]\hat{\tau}_0\\ &+\ [e^{-i\lambda/2}f(\omega-V/2)+e^{i\lambda/2}f(\omega+V/2)]\hat{\tau}_+\\ &-\ [(1-f(\omega-V/2))e^{i\lambda/2}+(1-f(\omega+V/2))e^{-i\lambda/2}]\hat{\tau}_-\,\}\ ,\end{aligned}\tag{52}$$

where $\hat{\tau}_i$ is the standard choice of Pauli matrices and $f(\omega)$ is the Fermi distribution function.

The correction to the distribution function due to the scattering term (47) is:

$$\begin{aligned}\delta_\alpha\ln\chi(\lambda) \ &= \ -\frac{\alpha^2}{4\pi^2\nu^2T_K^2}\sum_{p_1,p_2}(\varepsilon_{p_1}+\varepsilon_{p_2})^2\int_C dt_1dt_2g_{p_2}(t_2,t_1)g_{p_1}(t_1,t_2)\\ &= \ \frac{\alpha^2\mathcal{T}}{\pi T_K^2}\int d\omega\omega^2[(e^{-i\lambda}-1)n_L(1-n_R)+(e^{i\lambda}-1)n_R(1-n_L)]\ ,\end{aligned}\tag{53}$$

which, at zero temperature, contributes to Eq. (51) a term,

$$\delta_\alpha\ln\chi(\lambda)=\frac{\alpha^2V^3\mathcal{T}}{12\pi T_K^2}(e^{-i\lambda}-1)\ .\tag{54}$$

Regarding the correction to the charge distribution coming from the interaction term (48), any diagrams with a single insertion of the Green's function vanish (therefore there is also no $\alpha\phi$ cross term) and the only remaining connected graph yields:

$$\begin{aligned}\delta_\phi \ln\chi(\lambda) &= -\frac{\phi^2}{2\pi^2\nu^4 T_K^2}\int_C dt_1 dt_2 g(t_1,t_2)^2 g(t_2,t_1)^2 \qquad (55)\\ &= \frac{\phi^2}{\pi^2 T_K^2}\int_{-\infty}^{\infty} dt \frac{\cos^4[\lambda/2+(Vt)/2)]}{(t+i\alpha)^4}\\ &= \frac{\phi^2 V^3\mathcal{T}}{12\pi T_K^2}(e^{-i\lambda}-1)+\frac{\phi^2 V^3\mathcal{T}}{6\pi T_K^2}(e^{-2i\lambda}-1)\ .\end{aligned}$$

Combining the results we find that the zero–temperature charge distribution function is of the form:

$$\begin{aligned}\ln\chi(\lambda) &= iN\lambda + \frac{V^3\mathcal{T}}{12\pi T_K^2}\left[(\alpha^2+\phi^2)(e^{-i\lambda}-1)\right.\\ &+ \left. 2\phi^2(e^{-2i\lambda}-1)\right] + O(V^5)\ . \qquad (56)\end{aligned}$$

Let us now try to connect these results to the previous weak coupling calculation. The weak coupling expansion of the susceptibilities is given in Section III A. In the strong coupling limit we have:

$$\chi_e = (\Gamma\alpha)/(\pi T_K) \quad \chi_o = (\Gamma\phi)/(\pi T_K)\ , \qquad (57)$$

where in fact $\alpha=\phi=1$ and T_K is the Kondo temperature up to a pre-factor [22, 26, 27]. The programme of extending a Fermi liquid approach to non-equilibrium properties of the Anderson model has not been comprehensively carried out yet. There is a Fermi-liquid proof, due to Oguri [35], that the leading non-equilibrium correction to the zero–temperature current is of the form

$$I_{\rm bs} = \frac{V^3}{12\pi^2\Gamma^2}(\chi_e^2+5\chi_o^2)\ , \qquad (58)$$

which is valid for all U and interpolates between the weak–coupling and the strong–coupling regimes of the Anderson model. We see that the above result for $I_{\rm bs}$ is simply the strong–coupling limit of Oguri's formula. As to the noise and higher moments no analogous Fermi–liquid results exist, to the best of our knowledge. However, we would like put forward a *hypothesis* that Eq.(44) does represent such a generalisation. Indeed, we see that this formula is correct up to the U^2 order in weak coupling, it holds in the strong coupling limit, and it reproduces correctly the mean current at all orders in U.

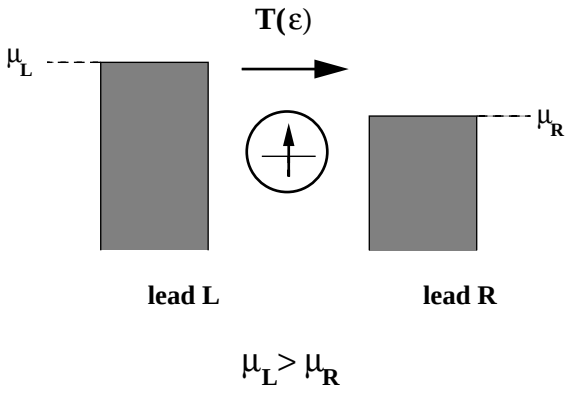

FIG. 1: A sketch of the distribution of particles in the leads when $\mu_L > \mu_R$, where μ_L and μ_R are the chemical potentials in the two leads.

IV. USING THE BETHE ANSATZ TO COMPUTE CURRENT NOISE

In the first part of this paper, we have focused on using Fermi liquid theory to compute the leading non-trivial correction (i.e. $\mathcal{O}(V^3)$) of the generating functional, $\chi(\lambda)$, for both strong and weak coupling. We have posited that this computation is exact and have performed a number of checks indicating that this is so. In the second part of the paper we adopt a different tact, instead focusing upon general features of the moments of the current (in particular the noise) in the strong coupling regime over a range of voltages (as measured in terms of T_K, the Kondo temperature). The tool we use to compute the noise is the Bethe ansatz.

Under the Landauer-Büttiker formalism [38], the DC noise, S, reduces to an expression given solely in terms of the transmission amplitude, T_σ, of electronic excitations across the dot:

$$S = \frac{e^2}{h} \sum_\sigma \int_{\mu_R}^{\mu_L} d\epsilon \; (T_\sigma - T_\sigma^2). \tag{59}$$

Here we imagine the chemical potential of the left lead is greater than that of the right lead and so the transmission amplitude, T_σ, governs the flow of electronic scattering states from the left to the right lead.

The Bethe ansatz gives us the ability to compute $T_\sigma(\epsilon)$ *in equilibrium* up to corrections of $\mathcal{O}(T_K/\sqrt{UT})$. We intend to use these equilibrium amplitudes to characterize *qualitatively* the behaviour of the noise as a function of $\mu_L - \mu_R$. Using equilibrium scattering amplitudes necessarily involves missing some of the physics present in a non-equilibrium setting.

Nonetheless we believe that this approach yields important insights. In particular, the general behaviour of the noise in a magnetic field that arises from this approach, should be robust enough to survive this particular approximation. This is already true of the current through the dot in a magnetic field where this approach [39, 43] yields results consistent with the observed enhancement [36, 47] in the conductance when $\mu_L - \mu_R \sim H$.

A. Computation of the Equilibrium $T_\sigma(\epsilon)$ Using the Bethe Ansatz

We now turn to a brief description of how to compute $T_\sigma(\epsilon)$ using the Bethe ansatz solution of the Anderson model. For additional details see [39]. The description comes in three parts. We first describe how to map the *equilibrium* (zero voltage) problem onto an one-channel Anderson model, a model directly solvable by Bethe ansatz. Although we work directly in a one lead formulation of the problem, we are still able to make contact with scattering in the original two lead picture. We so identify the relevant elements of the exact one-channel solution for computing scattering amplitudes.

To reformulate the problem in a way amenable to the Bethe ansatz, we introduce even/odd electrons

$$\psi_{e/o} = \frac{1}{\sqrt{\gamma_L^2 + \gamma_R^2}}(\gamma_L \psi_L \pm \gamma_R \psi_R). \tag{60}$$

With this, the odd electron decouples from the dot leaving us with an interacting theory of even electrons alone:

$$\begin{aligned} \mathcal{H} \;=\; \sum_\sigma \int dx \Big\{ & -i\psi^\dagger_{e\sigma}(x)\partial_x \psi_{e\sigma}(x) \\ & + (\gamma_L^2 + \gamma_R^2)^{1/2}\delta(x)(\psi^\dagger_{e\sigma}(x) d_\sigma + d^\dagger_\sigma \psi_{e\sigma}(x)) \Big\} \\ & + \Delta_0 \sum_\sigma n_\sigma + U n_\uparrow n_\downarrow . \end{aligned} \tag{61}$$

We point out that the Bethe ansatz could be directly applied to the two channel problem. However under the diagonalization of $\mathcal{H}$ carried out by the Bethe ansatz, the map (60) is implemented implicitly. As such we prefer to make the change of basis directly.

Under this one-lead reformulation, we are still able to make contact with the scattering amplitudes of electronic excitations off the quantum dot. Let $T(\epsilon)/R(\epsilon)$ be the transmission/reflection amplitudes of electronic excitations of energy, ϵ, between leads in the original

two lead picture. On the other hand, the even/odd excitations will scatter off the dot with some pure phase, $\delta_e(\epsilon)/\delta_o(\epsilon)$. As the scattering in the odd channel is trivial, $\delta_o(\epsilon) = 0$. The two sets of amplitudes are related according to (60):

$$\begin{aligned} e^{i\delta_e(\epsilon)} &= R(\epsilon) + T(\epsilon); \\ e^{i\delta_o(\epsilon)} &= 1 = R(\epsilon) - T(\epsilon). \end{aligned} \tag{62}$$

Our focus will henceforth be on computing δ_e.

To determine $\delta_e(\epsilon)$, we employ an energetics argument of the sort used by N. Andrei in the computation of the magnetoresistance of impurities in a bulk metal [40]. Imagine adding an electron to the system. Through periodic boundary conditions, its momentum is quantized, $p = 2\pi n/L$. If the dot was absent, the quantization condition would be determined solely by the conditions in the bulk of the system and we would write, $p_{\text{bulk}} = 2\pi n/L$. Upon including the dot, this bulk momentum is shifted by a term scaling as $1/L$. The quantization condition is then rewritten as

$$p = 2\pi n/L = p_{\text{bulk}} + \delta_e(\epsilon)/L, \tag{63}$$

where L is the system's length. The coefficient of the $1/L$ term is identified with the scattering phase of the electron off the dot. We thus must compute the impurity momenta of excitations in the problem.

The Bethe ansatz solution of the one channel Anderson model was first described in [41] and [42]. As with any problem with an $SU(2)$ symmetry, the Bethe ansatz yields a set of quantization conditions describing two types of excitations, one parameterized by k and associated roughly with charge excitations, and one parameterized by λ and associated approximately with spin excitations:

$$\begin{aligned} e^{ik_jL + i\delta(k_j)} &= \prod_{\alpha=1}^{M} \frac{g(k_j) - \lambda_\alpha + i/2}{g(k_j) - \lambda_\alpha - i/2}; \\ \prod_{j=1}^{N} \frac{\lambda_\alpha - g(k_j) + i/2}{\lambda_\alpha - g(k_j) - i/2} &= -\prod_{\beta=1}^{M} \frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}, \end{aligned} \tag{64}$$

where

$$\delta(k) = -2\tan^{-1}(\Gamma/(k - \epsilon_d));$$

$$g(k) = (k - \epsilon_d - U/2)^2/2U\Gamma;$$

$$\Gamma \;=\; (V_1^2 + V_2^2). \tag{65}$$

Here N is the total number of particles in the system and M marks out the spin projection of the system, $2S_z = N - 2M$ (in zero magnetic field $M = N/2$).

The possible solutions to the above quantization conditions are manifold. However most are only significant at finite temperature. At zero temperature, the ground state of the system is formed solely from real k's (and then only if the magnetic field is non-zero) and from bound states of real λ's together with complex k's. Specifically, the ground state contains:

i) N $-$ 2M real k'_js;

ii) M real λ'_αs;

iii) and associated with each of the M λ'_αs are

two complex k′s, $\mathrm{k}^\alpha_\pm$, described by

$$g(k^\alpha_\pm) = g(x(\lambda_\alpha) \mp iy(\lambda_\alpha)) = \lambda_\alpha \pm i/2;$$

$$x(\lambda) = U/2 + \epsilon_d - \sqrt{U\Gamma}(\lambda + (\lambda^2 + 1/4)^{1/2})^{1/2};$$

$$y(\lambda) = \sqrt{U\Gamma}(-\lambda + (\lambda^2 + 1/4)^{1/2})^{1/2}. \tag{66}$$

These elementary excitations implement a spin-charge separation in the model with the k's representing the charge sector while the λ's represent the spin sector.

In the continuum limit, these excitations are described by smooth densities, $\rho(k)$ for the real k's and $\sigma(\lambda)$ for the λ's. Equations valid at the symmetric point of the Anderson model describing these densities can be derived in the standard fashion [41, 42]:

$$\begin{aligned}\rho(k) \;=\; & \frac{1}{2\pi} + \frac{\Delta(k)}{L} \\ & + g'(k)\int_{-\infty}^{\infty} dk' R(g(k) - g(k'))(\frac{1}{2\pi} + \frac{\Delta(k')}{L})\end{aligned}$$

$$-g'(k)\int_{-D}^{B} dk'\rho(k')R(g(k)-g(k'));$$

$$\sigma(\lambda) \;=\; \int_{-\infty}^{\infty} dk(\frac{\Delta(k)}{L}+\frac{1}{2\pi})s(\lambda-g(k))$$

$$-\int_{-D}^{B} dk\rho(k)s(\lambda-g(k)). \tag{67}$$

where L is the system size and

$$\Delta(k) \;=\; \partial_k\delta(k)/2\pi,$$

$$s(x) \;=\; \frac{1}{2\cosh(\pi x)}. \tag{68}$$

B marks out the 'Fermi-surface' of the k distribution. Between $-D$ (the bottom of the band) and B there is a sea of k excitations. The Fermi surface of the λ particles on the other hand is set at $-\infty$: at the symmetric point, the sea of λ excitations in the ground state extends from $\lambda=-\infty$ to $\lambda=\tilde{Q}$, where $\tilde{Q}$ is the bandwidth of the λ excitations. This is a crucial simplification which makes possible many of our closed-form results. For most purposes both bandwidths, D and $\tilde{Q}$, can be taken to be ∞.

The density equations neatly divide into bulk and impurity pieces via

$$\rho(k) \;\rightarrow\; \rho_{\rm bulk}(k)+\rho_{\rm imp}(k)/L;$$

$$\sigma(\lambda) \;\rightarrow\; \sigma_{\rm bulk}(\lambda)+\sigma_{\rm imp}(\lambda)/L. \tag{69}$$

The impurity densities of states contain all the information needed about degrees of freedom living on the quantum dot. The equations governing these densities are

$$\rho_{\rm imp}(k) = \Delta(k)+g'(k)\int_{-\infty}^{\infty} dk' R(g(k)-g(k'))\Delta(k')$$

$$-g'(k)\int_{-D}^{B} dk'\rho_{\rm imp}(k')R(g(k)-g(k'));$$

$$\sigma_{\rm imp}(\lambda) \;=\; \int_{-\infty}^{\infty} dk\Delta(k)s(\lambda-g(k))$$

$$-\int_{-D}^{B} dk\rho_{\rm imp}(k)s(\lambda-g(k)). \tag{70}$$

For example, the total numbers of spin $\uparrow$ and $\downarrow$ electrons living on the dot are

$$\begin{aligned} n_{d\uparrow} &= \int_{-\infty}^{\infty} d\lambda \sigma_{\rm imp}(\lambda) + \int_{-\infty}^{B} dk \rho_{\rm imp}(k) \\ &= \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{B} dk \rho_{\rm imp}(k); \\ n_{d\downarrow} &= \int_{-\infty}^{\infty} d\lambda \sigma_{\rm imp}(\lambda) \\ &= \frac{1}{2} - \frac{1}{2}\int_{-\infty}^{B} dk \rho_{\rm imp}(k). \end{aligned} \tag{71}$$

The latter equations for each of $n_{d\uparrow}$ and $n_{d\downarrow}$ are a result of simplifications at the symmetric point.

The energies and momenta of these excitations can be derived through well known techniques [39]. The energies are given by

$$\begin{aligned} \epsilon(k) &= k - \frac{H}{2} - 2\int d\lambda\ x(\lambda) s(\lambda - g(k)) \\ &\quad - \int_{-D}^{B} dk' g'(k') \epsilon(k') R(g(k) - g(k')); \\ \epsilon(\lambda) &= 2x(\lambda) - 2\int d\lambda' R(\lambda - \lambda') x(\lambda') \\ &\quad + \int_{-D}^{B} dk\ g'(k) \epsilon(k) s(g(k) - \lambda). \end{aligned} \tag{72}$$

The momenta are akin to the densities in that they divide into bulk and impurity pieces [39]. The bulk momenta are related directly to the energies via

$$\begin{aligned} \epsilon(k) &= p(k) - \frac{H}{2}; \\ \epsilon(\lambda) &= p(\lambda). \end{aligned} \tag{73}$$

The impurity momenta can be expressed in terms of the impurity density of states

$$\begin{aligned} \partial_k p_{\rm imp}(k) &= 2\pi \rho_{\rm imp}(k); \\ \partial_\lambda p_{\rm imp}(\lambda) &= 2\pi \sigma_{\rm imp}(\lambda). \end{aligned} \tag{74}$$

As already discussed, the impurity momenta are the quantities crucial to computing scattering phases. These relations will thus allow us to express the scattering phases in terms of integrals over the impurity density of states.

In order to determine the scattering phase of an electron (as opposed to a spin or charge excitation), we must specify how to glue together a spin and a charge excitation to form the electron. The situation is analogous to adding a single particle excitation in the attractive Hubbard model. Adding a single spin $\uparrow$ electron to the system demands that we add a real k (charge) excitation. But at the same time we create a hole at some λ in the spin distribution. The number of the available slots in the spin distribution is determined by the total number of electrons in the system. Adding an electron to the system thus opens up an additional slot in the λ-distribution. The electron scattering phase off the impurity is then the difference of the right-moving k-impurity momentum, $p_{\rm imp}(k)$, and the left-moving λ-hole impurity momentum $-p_{\rm imp}(\lambda)$:

$$\delta_e^{\uparrow} = p_{\rm imp}^{\uparrow} = p_{\rm imp}(k) + p_{\rm imp}(\lambda). \tag{75}$$

We must now consider how to choose k and λ.

As we are interested in the DC noise, we must compute scattering away from the Fermi surface. Thus if we are to compute the scattering of an excitation of energy, ϵ_{el}, we must choose the k and λ such that

$$\epsilon_{el} = \epsilon(k) - \epsilon(\lambda). \tag{76}$$

However this constraint does not uniquely specify a particular choice of (k, λ). We, in general, cannot lift this degeneracy. However at the symmetric point of the Anderson model, we can make an ansatz which has already proven to be successful in the computation of the finite temperature linear response conductance [39]. The behaviour of the electron scattering phase is determined by the impurity densities, $\rho_{\rm imp}$ and $\sigma_{\rm imp}$. At the symmetric point of the Anderson model, the scattering phase is expected to vary as $\sim T_k$, the Kondo temperature. Of the two impurity densities, only $\rho_{\rm imp}$ varies as T_k while $\sigma_{\rm imp}$ is controlled by the much larger scale, $\sqrt{U\Gamma}$. Thus in computing electronic scattering phases away from the Fermi surface at $T = 0$, it is natural to choose λ at its Fermi surface value, i.e. $\lambda = -\infty$ and vary k according to the energy in which we are interested. Specifically, we choose k such that

$$\epsilon(k) = \epsilon_{el}. \tag{77}$$

We thus have removed the ambiguity in the choice of (k, λ).

In making this ansatz we are effectively doing the following. Imagine an electron in the leads with some energy ϵ_{el}. We can imagine expanding this state in terms of the basis of Bethe ansatz states:

$$|el >= \sum_s c_{el,s}|s\rangle, \tag{78}$$

where the states $|s\rangle$ are *exact* eigenfunctions of the Hamiltonian. Although we only possess incomplete knowledge of this expansion, it would be a reasonable guess that it contains multiple terms. However our ansatz supposes only a single state contributes. But because of the hierarchy of scales, $T_k \ll \sqrt{U\Gamma}$, in the problem, we expect additional terms in the expansion of Eqn. (78) to have coefficients of $\mathcal{O}(T_k/\sqrt{U\Gamma})$.

Under this ansatz, the scattering phase of the spin $\uparrow$ electron at some energy, ϵ_{el}, above the Fermi surface is then

$$\begin{aligned} \delta_e^{\uparrow}(\epsilon_{el}) &= p_{\rm imp}(k) + p_{\rm imp}(\lambda = -\infty); \\ &= 2\pi \int_Q^{\tilde{Q}} d\lambda \sigma_{\rm imp}(\lambda) + 2\pi \int_{-D}^{k} dk' \rho_{\rm imp}(k'), \\ \epsilon(k) &= \epsilon_{el}. \end{aligned} \tag{79}$$

When the magnetic field, H, is 0, $\epsilon(k) > 0$ and we can only directly compute the scattering of spin $\uparrow$ electrons. However with $H > 0$, $\epsilon(k)$ takes on negative values and so we can also compute spin $\uparrow$ hole scattering. To add a spin $\uparrow$ hole with energy, $\epsilon_{hole} > 0$, we remove a k and a λ-hole in the spirit of our previous ansatz:

$$\begin{aligned} \epsilon(k) &= -\epsilon_{hole}; \\ \lambda &= -\infty. \end{aligned} \tag{80}$$

The scattering phase is then

$$\begin{aligned} \delta_{ho}^{\uparrow}(\epsilon_{hole}) &= p_{\rm imp}(k) + p_{\rm imp}(\lambda) \\ &= 2\pi \int_Q^{\tilde{Q}} d\lambda \sigma_{\rm imp}(\lambda) + 2\pi \int_{-D}^{k} dk' \rho_{\rm imp}(k'), \end{aligned}$$

$$\epsilon(k) = -\epsilon_{hole}. \tag{81}$$

So far we have computed the scattering of spin ↑ objects. To compute spin ↓ quantities, we employ the particle-hole transformation relating spin ↑ to spin ↓:

$$\begin{aligned} c_\uparrow^\dagger(k) &\rightarrow c_\downarrow(-k); \\ c_\downarrow^\dagger(k) &\rightarrow c_\uparrow(-k); \\ d_\uparrow^\dagger &\rightarrow d_\downarrow; \\ d_\downarrow^\dagger &\rightarrow d_\uparrow; \\ \epsilon_d &\rightarrow \epsilon_d, \end{aligned} \tag{82}$$

where the last line only follows at the symmetric point. With this transformation, we obtain

$$\begin{aligned} {\delta^\downarrow}_{el}(\epsilon_{el}) &= \delta_{ho}^\uparrow(\epsilon_{ho} = \epsilon_{el}); \\ {\delta^\downarrow}_{ho}(\epsilon_{ho}) &= \delta_{el}^\uparrow(\epsilon_{el} = \epsilon_{ho}). \end{aligned} \tag{83}$$

Our inability to directly compute spin ↓ scattering is a technical peculiarity of the Bethe ansatz [39].

B. Nature of the Approximation

We have now described how to compute the scattering amplitudes as a function of energy in equilibrium. The error in using these amplitudes in describing out-of-equilibrium quantities such as the current or the noise has two possible sources. The first source can be seen from the way the differing chemical potentials of the right and left reservoirs appear in the Hamiltonian:

$$\mathcal{H}_\mu = \mu_L \int dx\ \psi_L^\dagger(x)\psi_L(x) + \mu_R \int dx\ \psi_R^\dagger(x)\psi_R(x). \tag{84}$$

Under the map to the even and odd basis (60), this term becomes non-diagonal, coupling the even and odd sectors. This coupling between sectors, in turn, lifts the model's integrability,

the basis on which we compute $T_\sigma(\epsilon)$. One might have hoped [39] that the state of the system in non-equilibrium could still be characterized by using in-equilibrium data in a fashion analogous to the manner in which out-of-equilibrium quantum Hall edges can be characterized exactly [44]. While in analyzing the Hall edges, an even-odd transformation of bosonic degrees of freedom is employed, the Hall case is much simpler as the current as well as the coupling to the voltage bias, can be expressed directly in terms of the odd degree of freedom. But in the case of the Anderson model, the current and voltage involve both even and odd degrees of freedom. In mapping back to the left-right basis, some sort of breaking of integrability then occurs. We will see this explicitly when we compute the noise at zero magnetic field and compare it to the exact Fermi liquid results.

A second source of error in using the equilibrium scattering amplitudes concerns the manner in which we construct the electronic scattering states. As we have already indicated, there is a multiplicity of choices in how we construct the scattering states. The particular choice we employed was the simplest that met the requirement that the energy dependence of the scattering varies on the Kondo scale. However this choice is not unique and while it produced excellent results for the behaviour of the finite temperature linear response conductance, it may not be optimal for the description of out-of-equilibrium scattering.

V. COMPUTATION OF NOISE IN ZERO FIELD: COMPARISON BETWEEN FERMI LIQUID THEORY AND BETHE ANSATZ

In this section we compute the noise in zero magnetic field and compare it at small voltages to the Fermi liquid results from the first part of this article. To compute the noise, we imagine biasing the leads as in Figure 1 with $\mu_R < \mu_L$. For convenience we set μ_L to zero and $\mu_R \equiv \mu$. Then as discussed in the previous section, the noise is given by

$$S = \frac{e^2}{h}\int_\mu^0 d\epsilon \big(T_\uparrow(1-T_\uparrow) + T_\downarrow(1-T_\downarrow)\big). \tag{85}$$

At $H = 0$ spin $\uparrow$ and spin $\downarrow$ scattering are the same, i.e. $T_\downarrow = T_\uparrow$. As we are scattering electrons from lead R with energy $\epsilon < 0$ into lead L, we are equivalently interested in computing the scattering phase of a hole. In Section IV we demonstrated that at $H = 0$ we are able to compute the scattering amplitude of a spin $\downarrow$ hole. Specifically we have

$$T_\downarrow(\epsilon) \;=\; T_\uparrow(\epsilon) = \sin^2(\frac{\delta^\downarrow_{ho}(\epsilon)}{2});$$

$$\delta^{\downarrow}_{ho}(\epsilon) \;=\; 2\pi \int_{Q}^{\tilde{Q}} d\lambda \sigma_{\rm imp}(\lambda) + 2\pi \int_{-D}^{k} dk' \rho_{\rm imp}(k'),$$

$$\epsilon(k) = \epsilon. \tag{86}$$

In [39] we were able to evaluate these expressions in closed form:

$$\begin{aligned}
\delta^{\uparrow}_{ho}(\epsilon) \;&=\; \frac{3}{2}\pi - \sin^{-1}\left(\frac{1-\epsilon^2/\tilde{T}_k^2}{1+\epsilon^2/\tilde{T}_k^2}\right) \\
&\quad +2\sum_{n=0}^{\infty} \frac{1}{1+2n}\left(\frac{\epsilon\pi}{\sqrt{2U\Gamma}}\right)^{1+2n} \\
&\quad \times \int dk e^{-\pi g(k)(1+2n)} \mathrm{Re}[\Delta(ik)]; \\
\tilde{T}_k \;&\equiv\; \frac{2}{\pi} T_k = \frac{2}{\pi}\sqrt{\frac{U\Gamma}{2}} e^{-\pi\left(\frac{U}{8\Gamma}-\frac{\Gamma}{2U}\right)}.
\end{aligned} \tag{87}$$

The last equation gives the crossover scale, T_k, the Kondo temperature, in terms of the bare parameters of the model [45]. With this, $T_{\uparrow/\downarrow}$ equals

$$T_{\uparrow/\downarrow} = \frac{1}{1+\frac{\epsilon^2}{\tilde{T}_k^2}} + \mathcal{O}\left(\frac{T_k}{\sqrt{U\Gamma}}\right). \tag{88}$$

For typical realization of Kondo physics in quantum dots, i.e. [36, 37], the error term is insignificant. And so we compute the noise to be

$$S(\mu) = \frac{e^2}{h}\left(\frac{\mu}{1+\frac{\mu^2}{\tilde{T}_k^2}} - \tilde{T}_k \tan^{-1}\left(\frac{\mu}{\tilde{T}_k}\right)\right). \tag{89}$$

The quantity S/V is solely a function of the ratio V/T_k and is plotted against $I/|V|$ in Fig. 2. (Note that $\mu = eV$.) We see that as V/T_k is varied, the noise rapidly rises at first, peaks at $eV \approx -1.15 T_k$, and then begins to gradually decline.

This behaviour is closely related to the Kondo resonance in the spectral impurity density of states. As we express the scattering phases in terms of the impurity density of states, we probe the resonance. The width, w, of this resonance, $w \sim T_k$, corresponds to the energy scale at which we expect maximal noise.

The noise in this case can be reexpressed in terms of the current, I, and the differential conductance, G,

$$I(\mu) \;=\; -2\frac{e}{h}\tilde{T}_k \tan^{-1}\left(\frac{\mu}{\tilde{T}_k}\right);$$

$$G(\mu) \;=\; 2\frac{e^2}{h}\frac{1}{1+\frac{\mu^2}{T_k^2}}, \tag{90}$$

with the result

$$S(\mu) = \frac{1}{2}\mu G(\mu) + \frac{e}{2}I(\mu). \tag{91}$$

At small μ,

$$S(\mu<0) = -\frac{2e^2}{h}\frac{8\pi^2}{96}\mu^3 T_K^2.$$

We can compare this result with the Fermi liquid result

$$S^{FL}(\mu) = -\frac{2e^2}{h}\frac{5\pi^2}{96}\mu^3 T_K^2.$$

In making this comparison, there is a certain arbitrariness in how one defines T_K. This can be overcome by appealing to the finite temperature linear response conductance, $G(T)$ to fix the manner in which T_K is to be defined. In our conventions then, $G(T) = 2e^2/h(1-\frac{\pi^4}{16}\frac{T^2}{T_K^2})$. We see then that the Fermi liquid result differs from the result based upon the equilibrium Bethe ansatz scattering states by a factor of 5/8.

A similar difference can be found between the Fermi liquid and Bethe ansatz computation for the current. At small μ, the leading order correction to the current (of $\mathcal{O}(\mu^3)$) is given by

$$\delta I(\mu) = 2\frac{e}{h}\frac{\pi^2}{12}\frac{\mu^3}{T_K^2}. \tag{92}$$

This compares to the Fermi liquid result

$$\delta I^{FL}(\mu) = 2\frac{e}{h}\frac{\pi^2}{32}\frac{\mu^3}{T_K^2}. \tag{93}$$

We see that the Fermi liquid result is considerably smaller than that of the Bethe ansatz. This might well reflect the role of incoherent scattering processes that would be unaccounted for properly by using equilibrium scattering amplitudes.

Finally we consider the value of the effective charge, e^*, in the problem. This charge is given as a ratio of the noise to the backscattering current:

$$e^* = S/I_{bs}. \tag{94}$$

In the case of the Bethe ansatz, we find $e^* = e$, that is, we find the Johnson-Nyquist result for shot noise in the weak scattering limit. However in Fermi liquid theory, the effective

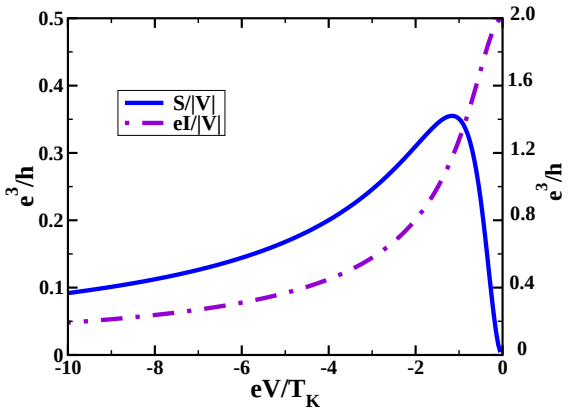

FIG. 2: Plot describing the evolution of both the noise, S/V, and the current, I/V, as a function of the applied voltage. The scale on the l.h.s. governs the noise while the scale on the r.h.s. governs the current.

charge is found to be $e^{*FL} = 5/3e$ [32]. That the effective charge goes unchanged from its non-interacting value is again presumably a consequence of the use of equilibrium scattering states.

While the use of the equilibrium Bethe ansatz scattering states gets quantitative details of the noise (and the current) incorrect, it gets qualitative features correct. In particular the conductance as a function of voltage is Lorenztian-like with a width of order T_K. Similarly the zero field noise as a function of voltage increases rapidly (on a scale of order T_K) and thereafter decreases slowly. Both of these features would be expected to be present based solely upon the presence of the Kondo/Abrikosov-Suhl resonance whose width is governed by the scale T_K.

VI. COMPUTATION OF NOISE IN FINITE FIELD

In this section we compute the noise in a finite magnetic (Zeeman) field. We will here argue that the finite field noise possesses features that can be considered a "smoking gun" [46] of Kondo physics. We believe that these features are robust and so should be captured by our approach.

To compute the noise,

$$S = \frac{e^2}{h}\int_{\mu}^{0} d\epsilon \big(T_{\uparrow}(1-T_{\uparrow}) + T_{\downarrow}(1-T_{\downarrow})\big), \tag{95}$$

in a finite field, we must consider the contributions of the spin $\uparrow$ and spin $\downarrow$ currents individually. From Section III, $T_\uparrow$ and $T_\downarrow$ are given by

$$\begin{aligned} T_\uparrow &= \sin^2(\frac{\delta^\uparrow_{ho}(\epsilon)}{2}); \\ \delta^\uparrow_{ho}(\epsilon) &= 2\pi\int_Q^{\tilde{Q}} d\lambda\sigma_{\rm imp}(\lambda) + 2\pi\int_{-D}^{k} dk'\rho_{\rm imp}(k'), \\ & \qquad \epsilon(k) = -\epsilon; \\ T_\downarrow &= \sin^2(\frac{\delta^\downarrow_{ho}(\epsilon)}{2}); \\ \delta^\downarrow_{ho}(\epsilon) &= 2\pi\int_Q^{\tilde{Q}} d\lambda\sigma_{\rm imp}(\lambda) + 2\pi\int_{-D}^{k} dk'\rho_{\rm imp}(k'), \\ & \qquad \epsilon(k) = \epsilon. \end{aligned} \tag{96}$$

In [39] we evaluated these expressions in two cases, $H \ll T_k$ and $H > T_k$:

case i: $H \ll T_k$

We found for spin $\uparrow$ hole scattering

$$\begin{aligned} \delta^\uparrow_{ho}(\epsilon_{ho} > 0) &= \frac{5}{4}\pi - \sin^{-1}\left(\frac{1-(\epsilon_{ho}-H)^2/\tilde{T}_k^2}{1+(\epsilon_{ho}-H)^2/\tilde{T}_k^2}\right) \\ &\quad + \frac{1}{2}\sin^{-1}\left(\frac{1-H^2/\tilde{T}_k^2}{1+H^2/\tilde{T}_k^2}\right), \end{aligned} \tag{97}$$

while for spin $\downarrow$ hole scattering we arrived at

$$\begin{aligned} \delta_{ho}(\epsilon_{ho} > 0) &= \frac{5}{4}\pi - \sin^{-1}\left(\frac{1-(\epsilon_{ho}+H/2)^2/\tilde{T}_k^2}{1+(\epsilon_{ho}+H/2)^2/\tilde{T}_k^2}\right) \\ &\quad + \frac{1}{2}\sin^{-1}\left(\frac{1-H^2/(4\tilde{T}_k^2)}{1+H^2/(4\tilde{T}_k^2)}\right). \end{aligned} \tag{98}$$

Consequently the transmission amplitudes in this case equal

$$T^\uparrow = \frac{1}{2}\left(1 + \frac{1+(H^2-\mu_2^2)/\tilde{T}_k^2}{(1+H^2/\tilde{T}_k^2)^{1/2}(1+(\mu_2+H)^2/\tilde{T}_k^2)}\right);$$

$$T^{\downarrow} \;=\; \frac{1}{2}\Bigg(1+\frac{1+(H^2/4-\mu_2^2)/\tilde{T}_k^2}{(1+H^2/4\tilde{T}_k^2)^{1/2}(1+(\mu_2-H/2)^2/\tilde{T}_k^2)}\Bigg). \tag{99}$$

case ii: $H > T_k$

Using a Weiner-Hopf analysis, $\delta_{ho}^{\uparrow}$ and $\delta_{ho}^{\downarrow}$ were determined in Ref. [39] to be

$$\delta_{ho}^{\uparrow} \;=\; \pi + 2\tan^{-1}(2(I^{-1}-g(k)));$$

$$\delta_{ho}^{\downarrow} \;=\; \frac{3\pi}{2} + \tan^{-1}(2(I^{-1}-b)), \tag{100}$$

where I^{-1} sets the Kondo scale, $T_k \sim e^{-\pi I^{-1}}$, and $I^{-1}-b$ is given in terms of the ratio H/T_k:

$$I^{-1} \;=\; \frac{U}{8\Gamma} - \frac{\Gamma}{2U};$$

$$I^{-1} - b \;=\; \frac{1}{\pi}\log\big(\frac{H}{2T_k}\sqrt{\frac{\pi e}{2}}\big). \tag{101}$$

k is parameterized in terms of the energy, ϵ, by the expression

$$\begin{aligned}\epsilon(k) \;=\; -H\Bigg(1 &- \frac{1}{2\pi}\tan^{-1}\frac{1}{g(k)-b} \\ &- \frac{1}{4\pi^2}\frac{1}{1+(g(k)-b)^2}\Bigg[\frac{\psi(1/2)}{\Gamma(1/2)}+1 \;-(g(k)-b)\tan^{-1}\big(\frac{1}{g(k)-b}\big) \\ &\qquad +\mathbf{C}+\frac{1}{2}\log(4\pi^2(1+(g(k)-b)^2))\Bigg]\Bigg) \\ &+\frac{\sqrt{2\Gamma U}}{\pi^2}\Bigg(\frac{1}{\sqrt{2e\pi}}\frac{e^{-b\pi}}{1+(g(k)-b)^2}+e^{-\pi g(k)}\tan^{-1}\big(\frac{1}{g(k)-b}\big)\Bigg) \\ &+\mathcal{O}((g(k)-b)^{-3})\end{aligned} \tag{102}$$

where $C = .577216\ldots$ is Euler's constant and b is given by

$$b = \frac{1}{\pi}\log\big(\frac{2}{H}\sqrt{\frac{U\Gamma}{\pi e}}\big). \tag{103}$$

With this

$$T^{\uparrow} \;=\; \frac{1}{1+(2(I^{-1}-g(k))^2)}+\cdots;$$

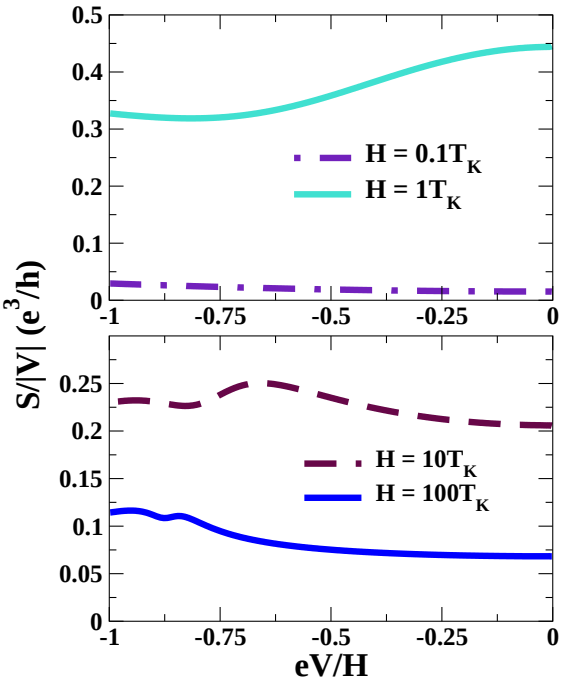

FIG. 3: Plots describing the behaviour of the noise, S/V, as a function of the applied voltage, V, for a variety of magnetic field values.

$$T^{\downarrow} = \frac{1}{2} - \frac{I^{-1} - b}{(1 + 4(I^{-1} - b)^2)^{1/2}} + \cdots . \tag{104}$$

In writing (104) we have omitted writing terms arising from the full expression for $\epsilon(k)$ in (102). But because of the logarithmic dependence upon H/T_k, such terms are needed if we are to compute the noise with reasonable accuracy for fields, H, not far in excess of T_k. Notice that the spin $\downarrow$ scattering does not vary as a function of energy, an approximation valid for $H \gg T_k$.

We plot the noise, $S/|V|$, in Figure 3 for a variety of values of H/T_k. For small H/T_k the noise is smooth and without structure. However as we vary H from $H < T_k$ to $H > T_k$, a marked variation occurs as seen in top panel of Figure 3. The noise rapidly increases, achieving its maximal value of roughly $H \sim T_k$, the crossover scale, before again decreasing. This behaviour is repeated in the differential noise $-\partial_V S$, plotted in Figure 4. The noise's maximal value at $H = T_k$ is a reflection of the maximum in $T^{\uparrow}(1 - T^{\uparrow})$. Such a maximum occurs for $T^{\uparrow} \sim 1/2$. Thus at $H = T_k$, the average transmission amplitude for spin $\uparrow$ excitations has been reduced roughly by $1/2$.

At large H/T_k, the noise develops a double humped structure near $e|V| \sim H$. This feature is more apparent when we examine the differential noise in the lower panel of Figure 4. In varying V near the peak, the differential conductance $G \propto T^{\uparrow}(V)$, passes through the value $G_{max}/2 = e^2/(2h)$ twice. As such the quantity, $\partial_V S \sim T^{\uparrow}(1-T^{\uparrow})$, possesses two peaks.

Given the bias at which it occurs, the doubled peak is intimately related to the peak in the differential conductance seen near $eV \sim H$. The peak in the differential conductance owes its origin to a field induced bifurcation in the Kondo resonance [46]: as the Kondo resonance shifts so does the peak in the conductance. In Refs. [39, 43] this bifurcation was studied where it was found that the peak occurs at a value of eV distinctly smaller than H and not $eV = H$. In the case of the noise, we again find that the peaked structure in it occurs at values of eV smaller than H.

We believe this double peaked structure in the noise, inasmuch as it depends on the gross dependency of the scattering amplitudes upon H, to be a robust feature. Less certain are the quantitative predictions that arise from this analysis. Nonetheless we will proceed to analyze the structure of the differential noise peaks. We do point out that we have some confidence that this analysis has merit as a subset of its corresponding predictions for the behavior of the *current* in a magnetic field have been shown to be at least qualitatively correct [47]. In particular, the prediction that the peak in the differential conductance occurs for values of eV smaller than H has been observed in experiments on carbon nanotubes [47].

The differential noise, $-\partial_V S$, is given by

$$\partial_V S = \frac{e^3}{h}\big(T^{\uparrow}(1-T^{\uparrow}) + T^{\downarrow}(1-T^{\downarrow})\big). \tag{105}$$

For $V \sim H$ and $H \gg T_k$, $T_{\downarrow} \ll T_{\uparrow}$, the locations of the peaks again occur when

$$T^{\uparrow} = 1/2.$$

As $T^{\uparrow} = \sin^2(\delta^{\uparrow}_{ho}/2)$, the scattering phases that correspond to this amplitude are, $\delta^{peak\uparrow}_{ho} = \frac{\pi}{2}/\frac{3\pi}{2}$. From (100), this in turn implies $g(k) = I^{-1} \pm 1/2$. Using 102, the biases, $V_{\pm}$, at which the two peaks occur equal

$$eV_{\pm} = -\frac{H}{2\pi}\frac{1}{I^{-1} - b \pm \frac{1}{2}}. \tag{106}$$

The height of the peaks can also be determined. The maxima of the peaks occur when $T^{\uparrow} = 1/2$. Consequently, the height of the peaks in $-\partial_V S$ are given by adding $e^3/(4h)$, the

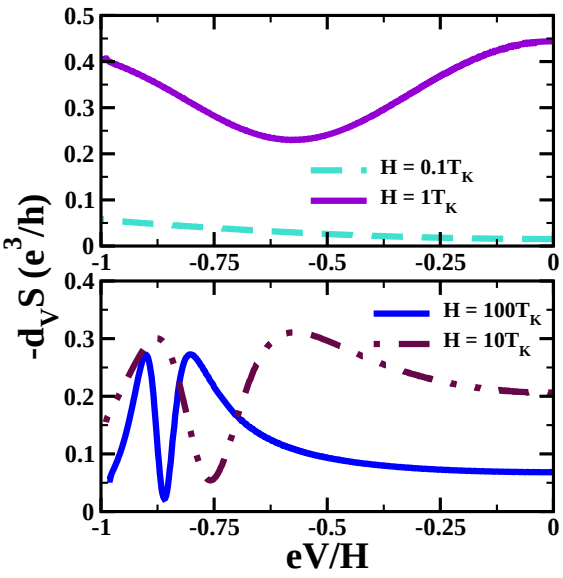

FIG. 4: Plots describing the behaviour of the differential noise, -$\partial_V S$, as a function of the applied voltage, V, for a variety of magnetic field values.

contribution due to the spin ↑ current, to the contribution from spin ↓ scattering with the result,

$$-\partial_V S^{\max} = \frac{e^3}{4h}\frac{4(I^{-1}-b)^2+2}{4(I^{-1}-b)^2+1}. \tag{107}$$

This result holds for either peak, a consequence of the lack of variation in $T^{\downarrow}$ for $eV, H \gg T_k$.

We are also able to compute the full width at half maximum (FWHM) of the peaks. As the peak maxima occur for $T^{\uparrow}(1-T^{\uparrow}) = 1/4$, or phases, $\delta_{ho}^{peak\uparrow} = \frac{\pi}{2}/\frac{3\pi}{2}$, the FWHM occur for $T^{\uparrow}(1-T^{\uparrow}) = 1/8$. For the peak, $\delta_{ho}^{peak\uparrow} = \pi/2$, the phases of the FWHM then correspond to

$$\delta_{ho}^{FWHM\uparrow} = \frac{\pi}{4},\ \frac{3\pi}{4}.$$

These two phases occur at energies parameterized by values of k given by

$$g(k) = I^{-1} + \frac{1}{2}\tan(\frac{3\pi}{8}/\frac{\pi}{8}). \tag{108}$$

Hence the width of this peak is

$$\begin{aligned} e\Delta V \ = \ & \frac{H}{2\pi}\Bigg(\tan^{-1}(\frac{1}{I^{-1}-b+\frac{1}{2}\tan(\frac{\pi}{8})}) \\ & - \tan^{-1}(\frac{1}{I^{-1}-b+\frac{1}{2}\tan(\frac{3\pi}{8})})\Bigg). \end{aligned} \tag{109}$$

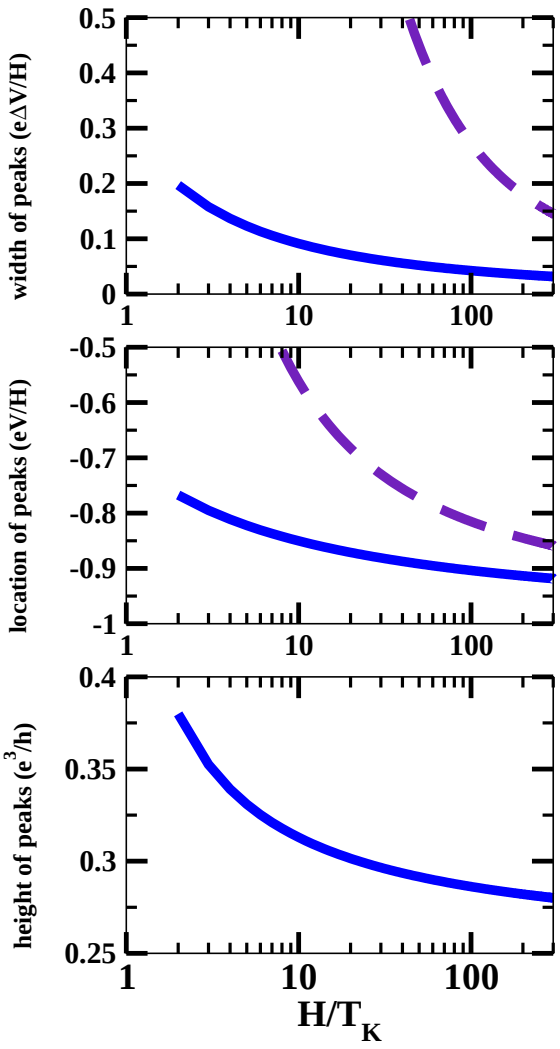

FIG. 5: Plots describing the evolution of the differential noise peaks with increasing magnetic field. In the top panel are plots of the widths of the two peaks, in the middle panel plots of the two peaks' locations, and in the bottom panel, a single plot of both peaks' (identical) height.

Similarly, for the peak corresponding to $\delta_{ho}^{peak\uparrow} = \frac{3\pi}{2}$, the phases of the half-maxima are

$$\delta_{ho}^{FWHM\uparrow} = \frac{5\pi}{4}, \ \frac{7\pi}{4},$$

and consequently, the FWHM of this peak is

$$e\Delta V = \frac{H}{2\pi}\Bigg(\tan^{-1}\Big(\frac{1}{I^{-1} - b - \frac{1}{2}\tan(\frac{3\pi}{8})}\Big) - \tan^{-1}\Big(\frac{1}{I^{-1} - b - \frac{1}{2}\tan(\frac{\pi}{8})}\Big)\Bigg). \tag{110}$$

The width of the two peaks is thus notably different, with the peak occurring nearer to $e|V| = H$ the narrower.

The various peak parameters are plotted in Figure 5. We see in the bottom panel that the height of the peaks approaches an asymptotic value of $e^3/4h$ logarithmically in H/T_k. This limit corresponds to a situation where only spin $\uparrow$ electrons contribute to the current. In the middle panel of Figure 5 are plotted the biases, $V_\pm$, at which the two peaks occur. In the large H/T_k limit, $eV_\pm$ approaches H. However even at large H/T_k there is a significant deviation from H, a consequence of the logarithmic dependence upon H/T_k. This behaviour mimics that of the peak in the differential conductance: for large H/T_k, this peak occurs at a value of eV distinctly smaller than H [39]. Finally in the top panel of Figure 5 is plotted the width of the two peaks. Again the peak that occurs at a bias closer to -H is the more narrow of the two.

VII. CONCLUSIONS

To summarize, we have presented analyzes of the moments of the current using both Fermi liquid perturbation theory and the Bethe ansatz. Both approaches however remain incomplete. In the Fermi liquid approach, while we have derived results for the higher moments of the current valid, we believe, at all orders of the interaction strength, we have done so only at the lowest non-trivial order in the voltage. To expand these computations to higher orders in the voltage remains a challenging problem for future research. In terms of the Bethe ansatz treatment, we have managed to identify various robust features of the current noise that should be experimentally identifiable in current realizations of quantum dots. In particular, we have identified a double peaked structure in the noise that appears at finite magnetic fields. But because of our use of *equilibrium* scattering amplitudes, we have been able to focus only upon qualitative aspects of the physics. Quantitatively, the methodology produces results at variance with Fermi liquid theory. It thus remains an important goal to pinpoint the precise origin of this disagreement.

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